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Solving forward-looking models of cross-country adjustment and monetary policy within the euro area

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Solving forward-looking models of cross-country adjustment within the euro area

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Abstract

This paper generalizes the standard methods of solving rational expectations models to the case of time-varying nonstochastic parameters, recurring in a finite cycle. Such a specification occurs in a simple stylized New Keynesian model of the euro area when we combine the rotation in the ECB Governing Council (as constituted by the Treaty of Nice) and home bias in the interest rate decisions taken by its members. In small and mid-size economies, this combination slightly increases output and inflation volatility, as compared to a monetary policy setup without rotation. The method of Christiano (2002) has also been applied to solve the model when we assume a lagged perception of foreign macroeconomic shocks by domestic agents. When the cross-country synchronization of shocks is low or moderate and when these shocks are relatively persistent, the exclusion of contemporaneous foreign shocks from domestic agents' information sets may raise the volatility of output. There is also some tentative evidence that this effect could particularly affect mid-size economies.

JEL Classification: C32, C61, E52, F15.

Keywords: EMU, monetary policy, solving rational expectations models, generalized Schur decomposition, heterogeneity.

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Contents

List of Tables

1 Introduction

The launch of the euro area project has opened space for empirical research and policy discussions on asymmetric shocks and adjustment mechanisms in their aftermath. Achieving a high capacity to absorb such shocks in the absence of autonomous monetary and exchange rate policy has become one of key economic policy targets, both for member and candidate countries. Market-based adjustment rests mainly upon the competitiveness channel (see European Commission, 2006, 2008; Narodowy Bank Polski, 2009). The adjustment process, however, may be hampered by the procyclical real interest rate mechanism („Walters critique"; see Walters, 1994).

The functioning of both mechanisms is highly dependent on the way in which economic agents in individual countries form their expectations. When a high inflation rate in an overheated economy translates into higher inflation expectations, then an asymmetric cyclical position – given weak (or no) reaction from the common central bank – results in a low real interest rate. This should additionally fuel economic activity, boost the cyclical amplitude and prolongue the period of adjustment. Nevertheless, if rational agents foresee that a protracted boom will undermine their country's external competitiveness, they anticipate the impact of deteriorated competitiveness. Consequently, they should reduce their expectations of future output gap and inflation rate, which weakens the real interest rate mechanism.

Expecations play therefore a key role in the functioning of both mechanisms and a thorough analysis of the adjustment dynamics is only possible when the expectation formation process is modelled with due precision. In this article, we apply a stylized, New Keynesian-based framework with rational expectations. With a hybrid specification of the IS and Phillips curves, it can – however – encompass various specifications of expectations being a linear combination of rationally expected values, past observations and a constant (e.g. adaptive or static).

A rational expectations model needs to be solved before use in simulation analyses. In this paper we argue that, under certain assumptions regarding monetary policy framework in the euro area and given the empirical evidence on expectations in the euro area from the literature, the application of classical solution methods (as in Blanchard and Kahn, 1980) can be insufficient.

Firstly, this is because the rotation scheme in the ECB Governing Council (henceforth: the Council), as constituted by the Treaty of Nice, may imply time-varying parameters in the Taylor rule approximating the ECB decisions. At present, the Council includes all the national central bank governors from the euro area countries with the right of vote in every decision meeting.[1] In this institutional setup, further euro area enlargement would imply a growing number of the Council members. This could lower the effectiveness of the decision process due to coordination problems (see e.g. Gerlach-Kristen, 2005). This was the motivation behind introducing a rotation system after the number of euro area members would exceed 15.[2] Under the Treaty, part of the governors would be rotationally excluded from the voting. As Subsection 2.2 presents, a time-varying model is adequate when the rotation is coupled with some home bias of the Council members in taking interest rate decisions.

Secondly, the inclination of economic agents to form inflation expectations first and above all on the basis of the events in the domestic economy justifies imposing heterogenous information sets across countries and hence across model equations. The simulation results presented in the paper suggest that both aspects can impact the volatility of inflation and output in the monetary union countries.

Section 2 describes the proposed model of a monetary union and discusses the limitations of standard solution methods in its case. Section 3 reviews the literature on solving rational expectations models, with particular attention being paid to the metod of Christiano (2002) applied here. In Subsection 3.2 a method of solving a model with variable coefficients is proposed. Section 4 presents the application of the methods considered. Section 5 concludes.

2 New Keynesian model of cross-country adjustment within the euro area

2.1 Adjustment mechanisms in the rational expectations model

The model considered in this paper draws heavily on the workhorse 3-equation New Keynesian model for monetary policy analyses. It is composed of an output gap equation (IS curve), inflation equation (Phillips curve) and nominal interest rate equation (central bank rule). The model has been extended to capture specific features of a group of open economies, forming a monetary union, with the competitiveness channel and the real interest rate effect.

The union-wide monetary policy is described by a Taylor rule with smoothing (see e.g. Sauer and Sturm, 2003, for an extensive survey on Taylor rule applications as approximations to the ECB policy):

[1]It also includes the ECB Board of Directors. For more details on the institutional context and the reform, see Narodowy Bank Polski (2009); Gorska (2009); Kosior et al. (2008); Szymczyk (2008).

[2]This is the case since January 1st, 2009 when Slovakia adopted the euro.

$$i_t = (1-\rho)\left[r^* + \pi^* + \gamma_\pi\left(\pi_t - \pi^*\right) + \gamma_y y_t\right] + \rho i_{t-1} \tag{1}$$

with i_t – nominal central bank rate at time t, y_t – output gap of the monetary union, π_t – inflation rate in the monetary union, r^* – natural interest rate, π^* – inflation target of the common central bank, $\rho \in (0;1)$ – smoothing parameter, $\gamma_\pi > 1$, $\gamma_y > 0$ – parameters for central bank reaction to deviation of inflation from the inflation target[3] and an open output gap respectively. The inflation rate and output gap in the entire monetary union are calculated as weighted averages over the member countries:

$$\pi_t = \sum_{j=1}^{n} w_j \pi_{j,t} \tag{2}$$

$$y_t = \sum_{j=1}^{n} w_j y_{j,t} \tag{3}$$

Country weights (vector $\mathbf{w}_{n\times1}$) reflect relative sizes of n economies ($j = 1, ..., n$) participating in the monetary union..[4]

The annualized inflation rate in country j (π_j) evolves according to a hybrid Phillips curve (see Galí and Gertler, 1999; Galí et al., 2001):

$$\pi_{j,t} = \omega_{f,j} E_t \pi_{j,t+1} + \omega_{b,j} \pi_{j,t-1} + \gamma_j y_{j,t} + \varepsilon^s_{j,t} \tag{4}$$

with $\varepsilon^s_{j,t}$ – cost-push shock in country j, $y_{j,t}$ – output gap in j. The path of the output gap is determined by the following IS curve, augmented with open economy components (see Clarida et al., 2001; Goodhart and Hofmann, 2005):

$$\begin{gathered} y_{j,t} = \beta_f E_t y_{j,t+1} + \beta_b y_{j,t-1} - \beta_r\left(i_t - E_t \pi_{j,t+1} - r^*_j\right) + \\ -\beta_c\left(P_{j,t} - P_{-j,t}\right) + \beta_s y_{-j,t} + \varepsilon^d_{j,t} \end{gathered} \tag{5}$$

where $y_{-j,t}$ denotes the output gap outside j, $P_{j,t}$ – log-level of prices in j, $P_{-j,t}$ – log-level of prices

[3] The condition $\gamma_\pi > 1$ is required for the Taylor rule (Taylor, 1993) to be fulfilled and the equilibrium to be determinate.

[4] Country weights used in the construction of Harmonized Index of Consumer Prices (HICP) for the euro area are derived from national accounts as the share of consumption spendings of households in a given country in the analogous value for the euro area. See for more: *Compendium of HICP reference documents*, Eurostat, http://epp.eurostat.ec.europa.eu/cache/ITY_OFFPUB/KS-AO-01-005/EN/KS-AO-01-005-EN.PDF.

outside j:

$$y_{-j,t} = \frac{\sum_{k,k\neq j} w_k y_k}{\sum_{k,k\neq j} w_k} \tag{6}$$

$$P_{j,t} = P_{j,t-1} + 0,25 \cdot \pi_{j,t} \tag{7}$$

$$P_{-j,t} = \frac{\sum_{k,k\neq j} w_k P_{k,t-1}}{\sum_{k,k\neq j} w_k} + \frac{\sum_{k,k\neq j} w_k \cdot 0.25 \cdot \pi_{k,t}}{\sum_{k,k\neq j} w_k} \tag{8}$$

The standard closed-economy specification has therefore been complemented with the real exchange rate divergence[5], $P_{j,t} - P_{-j,t}$, and external demand gap, $y_{-j,t}$. This corresponds to the point made by Clarida et al. (2001) that demand conditions in a small open economy are determined by external demand conditions and the ratio of domestic prices (expressed in foreign currency) to the world's price level. Excess appreciation undermines the price competitiveness of domestic goods abroad ($\beta_c > 0$), and foreign economic downturns translate into slowdowns at home ($\beta_s > 0$). The rest of the parameters in (4) and (5), in line with the New Keynesian literature, should be positive.

The model composed of equations (1)-(8) can be written in the form (13). The detailed description of matrix construction is provided in Appendix 1.

With standard assumptions, such as constant country weights in equations (2) and (3) as well as a standard expectation operator in (4) and (5), we can apply standard methods when solving the model for simulations (see Subsection 3.1). The following two subsections, however, will argue that these assumptions might have to be relaxed for the sake of an adequate description of the euro area economy.

2.2 Rotation scheme in the ECB Governing Council

The mandate of the Council is to maintain price stability in the entire euro area (see European Central Bank, 2003). When we interpret this literally, the conduct of monetary policy approximated by equations (1)-(3) would remain unaffected. However, the opponents of the voting system reform in the ECB claim that it is a step back in the european monetary integration that additionally emphasizes

[5] There is no nominal exchange rate dynamics between monetary union member countries, so the real exchange rate variance is only due to the difference in price log-levels.

the national structure of the Council (see Belke, 2003). Therefore, the counsequences of such a danger are worth considering.

Assume that every central bank governor implicitly prefers some nominal interest rate level, conditional upon the (possibly asymmetric) cyclical position of his country of origin:

$$i_{j,t} = (1-\rho)\left[r^* + \pi^* + \gamma_\pi\left(\pi_{j,t} - \pi_j^*\right) + \gamma_y y_{j,t}\right] + \rho i_{t-1} \tag{9}$$

If he or she wanted to reduce the cyclical stress in their country of origin (see Clarida et al., 1999; Calmfors, 2007), they would be inclined to vote in favour of interest rate changes towards $i_{j,t}$, even if these changes were at odds with (1).[6] The final preference of the national central bank governor, declared in the voting, is defined as a weighted average of the „pro-european" rate in (1) and the preferred rate for his country of origin, as in (9):

$$\tilde{i}_{j,t} = (1-\alpha)\, i_t + \alpha i_{j,t} \tag{10}$$

The parameter $\alpha \in [0;1]$ measures the home bias in the decision of the Council's members.[7] With fully „pro-european" voters, $\alpha = 0$. The other limiting case of fully home-biased voters occurs when $\alpha = 1$.

The outcome of voting at t is approximated by the arithmetic average over preferences submitted by the governors allowed to vote at t. After the reform, only 15 (of a higher number of) Council members would vote at a single meeting. Let $a_{j,t}$ be a dummy equal 1 when country j representative has got the right to vote at t and 0 otherwise. With these assumptions, the final interest rate decision of the ECB can be written as:

$$\bar{i}_t = \frac{1}{\sum_j a_{j,t}} \sum_{j=1}^{n} a_{j,t} \cdot \left((1-\alpha)\, i_t + \alpha i_{j,t}\right) \tag{11}$$

Substituting (1)-(3) and (9) into (11), we obtain the final form of the Taylor rule for the ECB:

[6] Clarida et al. (1999) differentiate between a cyclical and a structural (long-term) stress. Calmfors (2007) and Flaig and Wollmershäuser (2007) find some evidence that many euro area countries have suffered from the latter art of stress in the period 1999-2006. Nonetheless, the inclusion of structural stress into the governors' preferences would result in a „wandering" steady state of the model. In consequence, it is not impossible to analyze the second-moment properties of variables with the tools applied in this paper, but this extension would make the results dependent on the estimates of long-term inflation differentials (Harrod-Balassa-Samuelson effect) and natural rates of interest. This is why we leave the structural aspect of the stress for future empirical research.

[7] In this paper, we assume equal α across all Council members.

$$
\begin{aligned}
\bar{i}_t &= (1-\rho)\{r^* + \pi^* + \\
&+ \left[(1-\alpha)\,\mathbf{w}^T + \alpha\mathbf{a}_t^T\right]\gamma_\pi\left(\boldsymbol{\pi_t} - \boldsymbol{\pi^*}\right) + \left[(1-\alpha)\,\mathbf{w}^T + \alpha\mathbf{a}_t^T\right]\gamma_y\mathbf{y_t}\} + \\
&+ \rho i_{t-1}
\end{aligned} \tag{12}
$$

where symbols in bold subscripted t are vectors of size $n \times 1$ containing a sequence of identically denoted variables over countries, and $\boldsymbol{\pi}^* = \pi^* \cdot \mathbf{1}_{n\times1}$. Note that in (12) the parameters for inflation rates and output gaps in individual economies vary in time. In consequence, so does the matrix $\mathbf{B}$ in (13).

Non-constant parameters of the model (13) prevent us from applying standard solution methods described in Subsection 3.1.

2.3 Heterogeneity in formation of expectations

A vast battery of literature analyses the cross-country heterogeneity of the euro area and its consequences for the common monetary policy conduct. The most often explored research fields include differences in product and labour market flexibility (HM Treasury, 2003; Rumler, 2007), inflation persistence (Benigno and Salido, 2006), monetary transmission mechanisms (Clausen and Hayo, 2006) or business cycle synchronization (Skrzypczynski, 2006). A key dimension of heterogeneity are also expectation formation mechanisms.

There is empirical evidence in favour of this heterogeneity. The estimates of Taylor rule parameters for countries that formed the euro area in 1999 suggest that individual central banks conducted monetary policy in significantly different manners before they finally passed this responsibility to the ECB (Eleftheriou et al., 2006). Berger et al. (2006) find econometric evidence that expectations of future ECB decisions substantially varied in the geografic dimension in the first years of the euro area. Woodford (2006) argues that the process of learning the new monetary policy regime among economic agents and hence altering their expectation formation habits might be protracted.

In a monetary union, agents could expect the foreign macroeconomic shocks to hit their domestic economy via a few channels. Firstly, the common central bank would react to foreign demand shocks with a move in the common policy rate, which would in turn translate directly into change in domestic monetary conditions. Secondly, a foreign shock affects future price dynamics abroad. As a result, the real exchange rate would change – even when there were no direct price effects at home – which is another way to influence the domestic monetary conditions. Thirdly, foreign business cycle affects the domestic output due to international trade and investment links. Economic agents are therefore capable to predict an economic slowdown at home when they observe one in other countries.

Outside a monetary union, agents would have less incentive to monitor the foreign events. Firstly, the reaction of foreign central banks to foreign shocks does not automatically affect the nominal interest rates at home, which remain under the command of the domestic central bank. Secondly, a shock affecting foreign price dynamics does not necessarily translate into a shift in competitive position of domestic versus foreign producers, as measured with the real exchange rate. More precisely, this rate is also dependent on the nominal exchange rate, which can absorb asymmetric shocks (see Stazka, 2008). Because of these two channels, Marzinotto (2008) argues e.g. that mid-size economies are largely at risk of excessive wage growth. Namely, their trade unions are too small to influence the ECB decisions in a significant way and too large not to fear the loss of external competitiveness. Thirdly, the ample literature on the endogeneity of OCA criteria (e.g. integration of finance and trade in a common currency area, see Frankel and Rose, 1998 or Narodowy Bank Polski, 2009 for a survey) suggests growing interdependence of individual countries' output gaps after creating a monetary union.

For the reasons listed above, domestic agents – especially at the initial stages of participation in a monetary union – can be accustomed to form their expectations mainly on the basis of domestic events and to a lesser extent on the basis of foreign shocks. This would break the underlying assumptions of the standard expectations operator applied in the model (13), based on a common information set of agents in each country. As a consequence, this aspect of heterogeneity must be analyzed beyond the standard model solution methods.

3 Solving linear rational expectations models

3.1 Literature overview

A dynamic linear model with rational expectations can be written as (Blanchard and Kahn, 1980):

$$\mathbf{A}E_t\mathbf{x}_{t+1} = \mathbf{B}\mathbf{x}_t + \mathbf{C}\boldsymbol{\varepsilon}_t \tag{13}$$

with $\mathbf{x}_t$ – vector of variables at time t, $\boldsymbol{\varepsilon}_t$ – vector of random disturbances, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ – matrices of model parameters. The solution of the model is a transformation of (13) into a recursive law of motion (see Blanchard and Kahn, 1980; Uhlig, 1999; Klein, 2000; Sims, 2001):

$$\mathbf{x}_t = \mathbf{M}\mathbf{x}_{t-1} + \mathbf{N}\boldsymbol{\varepsilon}_t \tag{14}$$

This transformation is usually performed to run counterfactual simulations, impulse-response functions and second moment analysis (DeJong and Dave, 2007; Christiano, 2002). Lindé (2005), Fuhrer and Rudebusch (2004) and other authors stress the possibility to use the mapping to estimate the parameters of the solved model (14) directly via full information maximum likelihood method. They show with Monte Carlo simulation exercises that such estimation outperforms the GMM estimator, traditionally applied in empirical investigations of the New Keynesian model.

Blanchard and Kahn (1980) solved the model (13) assuming nonsingularity of $\mathbf{A}$ and performing the Jordan decomposition on $\mathbf{A}^{-1}\mathbf{B}$. They also developed a milestone theorem about the existence and uniqueness of a solution: the number of variables predetermined at t included in $\mathbf{x}_t$ must equal the number of eigenvalues of the matrix $\mathbf{A}^{-1}\mathbf{B}$ that do not exceed 1 in absolute value (saddle-path stability).

The assumption of a nonsingular matrix $\mathbf{A}$ is relaxed by Klein (2000) as he applies generalized complex Schur decomposition for matrices $\mathbf{A}$ and $\mathbf{B}$. A similar method is proposed by Sims (2001) for the model (13) expressed in terms of true future values of variables and expectation errors rather than the expectation operator. The solution results from a unique linear mapping from $\boldsymbol{\varepsilon}_t$ to the expectation errors. Söderlind (1999) applies a Klein-based algorithm and admits that generalized eigenvalues equal 1 in absolute terms can be classified as stable when the vector $\mathbf{x}_t$ contains variables that are explicitly nonstationary by construction (which also is the case in our model). Uhlig (1999) proposes a method of undetermined coefficients, which reduces the problem to solving a matrix quadratic equation.

Christiano (2002) proposed a modification of the latter method applicable when individual equations in the system have various information sets associated with their expectational terms. This approach is useful when different groups of economic agents take their decisions with heterogenous knowledge of contemporaneous values of economic shocks. Nonlinear models are solved with numerical methods (see DeJong and Dave, 2007 for an overview).

The solutions of models with variable coefficients proposed in the literature are usually designed for stochastic parameters (e.g. Markov switching with finite number of states). Farmer et al. (2008) apply a minimum state variable solution that expands the vector x_t times the number of possible states and adjust the parameter matrices appropriately.

3.2 Model with time-varying parameters

The time-varing Taylor rule (12) requires solving a model like (13), but with time-varying parameters:

$$\mathbf{A}_{(t)} E_t \left(\mathbf{x}_{t+1}\right) = \mathbf{B}_{(t)} \mathbf{x}_t + \mathbf{C}_{(t)} \mathbf{f}_t \tag{15}$$

The solution proposed below builds upon the algorithm by Klein (2000)[8], while introducing some necessary generalizations. It is useless to start with a single generalized Schur decomposition because the factor matrices we would obtain inherit the nonconstancy and parameter matrices for $\mathbf{x}_t$ and $E_t\left(\mathbf{x}_{t+1}\right)$ would not be upper triangular as we need.[9] Instead, we exploit the assumption that $\mathbf{A}_{(t)}$ and $\mathbf{B}_{(t)}$ vary in time, but the values recur after m periods, i.e. $\mathbf{A}_{(t+j)} = \mathbf{A}_{(t+j+i\cdot m)}$ and $\mathbf{B}_{(t+j)} = \mathbf{B}_{(t+j+i\cdot m)}$ for each $j = 0, 1, ..., m-1$ and each $i \in \mathbb{N}$. Let us first factorize the matrices $\mathbf{A}_{(t)}$ and $\mathbf{B}_{(t)}$ using a sequence of generalized complex Schur decompositions:

$$\begin{aligned} \mathbf{Q}_{(t)} \mathbf{A}_{(t)} \mathbf{Z}_{(t)} &= \mathbf{S}_{(t)} \\ \mathbf{Q}_{(t)} \mathbf{B}_{(t)} \mathbf{Z}_{(t)} &= \mathbf{T}_{(t)} \end{aligned} \tag{16}$$

with $\mathbf{S}$ and $\mathbf{T}$ – upper triangular matrices, $\mathbf{Q}$ and $\mathbf{Z}$ – unitary matrices ($\mathbf{Q}\mathbf{Q}^H = \mathbf{Q}^H\mathbf{Q} = \mathbf{Z}\mathbf{Z}^H = \mathbf{Z}^H\mathbf{Z} = \mathbf{I}$).[10] For the decomposition to be unique, we impose a restriction that diagonal elements of $\mathbf{S}$ and $\mathbf{T}$ are ordered in such a way that generalized eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ (equal $\frac{S_{i,i}}{T_{i,i}}$) ascend with rising index i.

Using (16) we can rewrite (15) for each t as:

$$\mathbf{S}_{(t)} \mathbf{Z}_{(t)}^H E_t \mathbf{x}_{t+1} = \mathbf{T}_{(t)} \mathbf{Z}_{(t)}^H \mathbf{x}_t + \mathbf{Q}_{(t)} \mathbf{C}_{(t)} \mathbf{f}_t \tag{17}$$

Let us write the equation for t, $t+1$, ..., $t+m-1$ and solve each of them for $\mathbf{x}$:

[8]In our model, it would suffice to allow for only one time-varying matrix – either A or B, as with the assumptions from section 2.2 we could place all time-varying parameters into a single matrix. However, it does not really simplify further derivations as the generalized Schur decomposition with imposed eigenvalue ordering is unique and all the output matrices would inherit time-variability, no matter how many input matrices (1 or 2) would bear this feature. Neither does time-dependent $\mathbf{C}_{(t)}$ cause any significant analytical or numerical complication. This is why the system (15) and its solution is expressed in more general terms that our example would require.

[9]A time-varying matrix $\mathbf{Z}_{(t)}$ (see (16)) does not permit us to define the substitution (23) in a unique way. If we chose some arbitrary $\mathbf{Z}$ matrix in time (say, $\mathbf{Z}_{(t)}$), every equation would link two different variables, which would obviously leave no space to proceed. This problem would indeed be solved by finding the generalized Schur decomposition for $\mathbf{A}_{(t)}$ and $\mathbf{B}_{(t+1)}$. However, in the latter case, there would be no $\mathbf{Q}_{(t)}$ to premultiply any equation leaving both matrices in question upper triangular (see (16)).

[10]Superscript H denotes hermitian transpose. $\mathbf{S}$,$\mathbf{T}$,$\mathbf{Q}$ and $\mathbf{Z}$ are complex matrices.

$$
\begin{aligned}
\mathbf{x}_t = \quad & \mathbf{Z}_{(t)}\mathbf{T}_{(t)}^{-1}\mathbf{S}_{(t)}\mathbf{Z}_{(t)}^{H}E_t\left(\mathbf{x}_{t+1}\right)-\mathbf{Z}_{(t)}\mathbf{T}_{(t)}^{-1}\mathbf{Q}_{(t)}\mathbf{C}_{(t)}\mathbf{f}_t \\
\mathbf{x}_{t+1} = \quad & \mathbf{Z}_{(t+1)}\mathbf{T}_{(t+1)}^{-1}\mathbf{S}_{(t+1)}\mathbf{Z}_{(t+1)}^{H}E_{t+1}\left(\mathbf{x}_{t+2}\right)-\mathbf{Z}_{(t+1)}\mathbf{T}_{(t+1)}^{-1}\mathbf{Q}_{(t+1)}\mathbf{C}_{(t+1)}\mathbf{f}_{t+1} \\
\vdots \quad & \\
\mathbf{x}_{t+m-1} = \quad & \mathbf{Z}_{(t+m-1)}\mathbf{T}_{(t+m-1)}^{-1}\mathbf{S}_{(t+m-1)}\mathbf{Z}_{(t+m-1)}^{H}E_{t+m-1}\left(\mathbf{x}_{t+m}\right)+ \\
& -\mathbf{Z}_{(t+m-1)}\mathbf{T}_{(t+m-1)}^{-1}\mathbf{Q}_{(t+m-1)}\mathbf{C}_{(t+m-1)}\mathbf{f}_{t+m-1}
\end{aligned}
\tag{18}
$$

A bottom-up sequence of substitutions and the law of iterated expectations (see Ljungqvist and Sargent, 2004) allows us to write an equation for $\mathbf{x}_t$:[11]

$$
\begin{aligned}
\mathbf{x}_t \quad = \quad & \underbrace{\left[\prod_{i=0}^{m-1}\mathbf{Z}_{(t+i)}\mathbf{T}_{(t+i)}^{-1}\mathbf{S}_{(t+i)}\mathbf{Z}_{(t+i)}^{H}\right]}_{\mathbf{D}_{(t)}}E_t\left(\mathbf{x}_{t+m}\right)+ \\
& -\underbrace{\left\{\left[\sum_{k=1}^{m-1}\left(\prod_{l=1}^{k}\mathbf{Z}_{(t+l-1)}\mathbf{T}_{(t+l-1)}^{-1}\mathbf{S}_{(t+l-1)}\mathbf{Z}_{(t+l-1)}^{H}\right)\mathbf{Z}_{(t+k)}\mathbf{T}_{(t+k)}^{-1}\mathbf{Q}_{(t+k)}\mathbf{C}_{(t+k)}E_t\mathbf{f}_{t+k}\right]+\mathbf{Z}_{(t)}\mathbf{T}_{(t)}^{-1}\mathbf{Q}_{(t)}\mathbf{C}_{(t)}\mathbf{f}_t\right\}}_{\Sigma_{k=0}^{m-1}\mathbf{R}_{\mathbf{k}(t)}E_t\mathbf{f}_{t+k}}
\end{aligned}
\tag{21}
$$

Once again, we perform a complex generalized Schur decomposition of $\mathbf{D}_{(t)}$ and $\mathbf{I}$ (as the parameter matrix for $\mathbf{x}_t$):

$$
\begin{aligned}
\mathbf{Q}_{(t)}\mathbf{D}_{(t)}\mathbf{Z}_{(t)} &= \mathbf{S}_{(t)} \\
\mathbf{Q}_{(t)}\mathbf{I}\mathbf{Z}_{(t)} &= \mathbf{T}_{(t)}
\end{aligned}
\tag{22}
$$

with the usual restriction on ordering generalized eigenvalues. Let us define an auxiliary variable:

[11] It would be much simpler to obtain (21) at the cost of some loss of generality. Assume nonsingular $\mathbf{B}_{(t)}^{-1}$ for each t (this is i.a. the case in the example considered here) and solve m first equations for $\mathbf{x}$:

$$
\begin{aligned}
&\mathbf{x}_t = \mathbf{B}_{(t)}^{-1}\mathbf{A}_{(t)}E_t\left(\mathbf{x}_{t+1}\right)-\mathbf{B}_{(t)}^{-1}\mathbf{C}_{(\mathbf{t})}\mathbf{f}_t \\
&\mathbf{x}_{t+1} = \mathbf{B}_{(t+1)}^{-1}\mathbf{A}_{(t+1)}E_{t+1}\left(\mathbf{x}_{t+2}\right)-\mathbf{B}_{(t+1)}^{-1}\mathbf{C}_{(t+1)}\mathbf{f}_{t+1} \\
&\vdots \\
&\mathbf{x}_{t+m-1} = \mathbf{B}_{(t+m-1)}^{-1}\mathbf{A}_{(t+m-1)}E_{t+1}\left(\mathbf{x}_{t+m}\right)-\mathbf{B}_{(t+m-1)}^{-1}\mathbf{C}_{(t+m-1)}\mathbf{f}_{t+m-1}
\end{aligned}
\tag{19}
$$

A sequence of substitutions similar to (18) and iterated expectations yield the following equation, equivalent to (21):

$$
x_t = \underbrace{\mathbf{B}_{(t)}^{-1}\mathbf{A}_{(t)}\mathbf{B}_{(t+1)}^{-1}\mathbf{A}_{(t+1)}\ldots\mathbf{B}_{(t+m-1)}^{-1}\mathbf{A}_{(t+m-1)}}_{\mathbf{D}_{(t)}}E_t\left(\mathbf{x}_{t+m}\right)-\underbrace{\mathbf{B}_{(t)}^{-1}\left[\left(\left(\sum_{k=1}^{m-1}\right)\left(\prod_{l=1}^{k}\mathbf{A}_{(t+l-1)}\mathbf{B}_{(t+l)}^{-1}\right)\mathbf{C}_{(t+k)}E_t\mathbf{f}_{t+k}\right)+\mathbf{C}_{(t)}\mathbf{f}_t\right]}_{\Sigma_{k=0}^{m-1}\mathbf{R}_{\mathbf{k}(t)}E_t\mathbf{f}_{t+k}}
\tag{20}
$$

$$\tilde{\mathbf{x}}_t = \mathbf{Z}_{(t)}^H \mathbf{x}_t \tag{23}$$

In line with the conventional treatment in the literature, let $\mathbf{x}_t$ be ordered in such a way that the first partition ($\mathbf{x}_{\mathbf{1},t}$) contains variables predetermined at t. Analogous partitioning of $\tilde{\mathbf{x}}_t$, substitution of (22) and (23) into (21), premultiplication by $\mathbf{Q}_{(t)}$ and conformable partitioning of $\mathbf{S}_{(t)}$, $\mathbf{T}_{(t)}$ and $\mathbf{Q}_{(t)}$ yield:

$$\begin{bmatrix} \mathbf{S}_{\mathbf{11}(t)} & \mathbf{S}_{\mathbf{12}(t)} \\ 0 & \mathbf{S}_{\mathbf{22}(t)} \end{bmatrix} E_t \begin{bmatrix} \tilde{\mathbf{x}}_{\mathbf{1},t+m} \\ \tilde{\mathbf{x}}_{\mathbf{2},t+m} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{\mathbf{11}(t)} & \mathbf{T}_{\mathbf{12}(t)} \\ 0 & \mathbf{T}_{\mathbf{22}(t)} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_{\mathbf{1},t} \\ \tilde{\mathbf{x}}_{\mathbf{2},t} \end{bmatrix} + \begin{bmatrix} \mathbf{Q}_{\mathbf{1}(t)} \\ \mathbf{Q}_{\mathbf{2}(t)} \end{bmatrix} \left(\Sigma_{k=0}^{m-1} \mathbf{R}_{\mathbf{k}(t)} E_t \mathbf{f}_{t+k}\right) \tag{24}$$

Following Klein (2000), we solve the lower, decoupled row of (24) for $\tilde{\mathbf{x}}_{2,t}$:

$$\tilde{\mathbf{x}}_{\mathbf{2},t} = \mathbf{T}_{\mathbf{22}(t)}^{-1} \mathbf{S}_{\mathbf{22}(t)} E_t \tilde{\mathbf{x}}_{2(t),t+m} - \mathbf{T}_{\mathbf{22}(t)}^{-1} \mathbf{Q}_{\mathbf{2}(t)} \left(\Sigma_{k=0}^{m-1} \mathbf{R}_{\mathbf{k}(t)} E_t \mathbf{f}_{t+k}\right) \tag{25}$$

The finite cycle of length m, in which the parameters of $\mathbf{A}_{(t)}$ and $\mathbf{B}_{(t)}$ recur, implies $\mathbf{D}_{(t)} = \mathbf{D}_{(t+m)}$ and $\mathbf{R}_{\mathbf{k}(t)} = \mathbf{R}_{\mathbf{k}(t+m)}$ for each k. We can therefore shift (21) m periods forward without changing the parameters:

$$\mathbf{x}_{t+m} = \mathbf{D}_{(t)} E_{t+m} \left(\mathbf{x}_{t+2m}\right) + \Sigma_{k=0}^{m-1} \mathbf{R}_{\mathbf{k}(t)} E_{t+m} \mathbf{f}_{t+m+k} \tag{26}$$

Matrices $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{S}$ and $\mathbf{T}$, resulting from the Schur decomposition of both matrices of interest in the above system, will equal those obtained in (22). Then, we can shift shift (25) by any multiple of m without changing the parameters:

$$\begin{aligned} \tilde{\mathbf{x}}_{2,t+m} &= \mathbf{T}_{\mathbf{22}(t)}^{-1} \mathbf{S}_{\mathbf{22}(t)} E_{t+m} \tilde{\mathbf{x}}_{2,t+2m} - \mathbf{T}_{\mathbf{22}(t)}^{-1} \mathbf{Q}_{\mathbf{2}(t)} \left(\Sigma_{k=0}^{m-1} \mathbf{R}_{\mathbf{k}(t)} E_{t+m} \mathbf{f}_{t+m+k}\right) \\ \tilde{\mathbf{x}}_{2,t+2m} &= \mathbf{T}_{\mathbf{22}(t)}^{-1} \mathbf{S}_{\mathbf{22}(t)} E_{t+2m} \tilde{\mathbf{x}}_{2,t+4m} - \mathbf{T}_{\mathbf{22}(t)}^{-1} \mathbf{Q}_{\mathbf{2}(t)} \left(\Sigma_{k=0}^{m-1} \mathbf{R}_{\mathbf{k}(t)} E_{t+2m} \mathbf{f}_{t+2m+k}\right) \\ &\vdots \end{aligned} \tag{27}$$

As in (19), a sequence of substitutions in (25) and (27) and iterating expectations allows us to express $\tilde{\mathbf{x}}_{\mathbf{2},t}$ as an infinite sum:

$$\tilde{\mathbf{x}}_{\mathbf{2},t} = -\sum_{i=0}^{+\infty}\left\{\left(\mathbf{T}^{-1}_{\mathbf{22}(t)}\mathbf{S}_{\mathbf{22}(t)}\right)^{i}\mathbf{T}^{-1}_{\mathbf{22}(t)}\mathbf{Q}_{\mathbf{2}(t)}\left(\Sigma_{k=0}^{m-1}\mathbf{R}_{\mathbf{k}(t)}E_t\mathbf{f}_{t+i\cdot m+k}\right)\right\} \tag{28}$$

At this point, we need to know the expected path of future random disturbances, conditional on the information that agents have at t.[12] In rational expectations models, autoregressive error terms are natural by construction and hence commonly applied, so let us assume a VAR representation (see e.g. Mavroeidis, 2005):

$$\mathbf{f}_t = \mathbf{\Phi}\mathbf{f}_{t-1} + \boldsymbol{\varepsilon}_t \tag{29}$$

With $E_t\boldsymbol{\varepsilon}_{t+k} = 0$, $k = 1, 2, ...$, we can write the infinite sum (28) as

$$\begin{aligned}
\tilde{\mathbf{x}}_{\mathbf{2},t} &= -\sum_{i=0}^{+\infty}\left[\left(\mathbf{T}^{-1}_{\mathbf{22}(t)}\mathbf{S}_{\mathbf{22}(t)}\right)^{i}\mathbf{T}^{-1}_{\mathbf{22}(t)}\mathbf{Q}_{\mathbf{2}(t)}\left(\Sigma_{k=0}^{m-1}\mathbf{R}_{\mathbf{k}(t)}\mathbf{\Phi}^{i\cdot m+k}\mathbf{f}_t\right)\right] = \\
&= -\sum_{i=0}^{+\infty}\left[\left(\underbrace{\mathbf{T}^{-1}_{\mathbf{22}(t)}\mathbf{S}_{\mathbf{22}(t)}}_{\mathbf{F}_{(t)}}\right)^{i}\underbrace{\mathbf{T}^{-1}_{\mathbf{22}(t)}\mathbf{Q}_{\mathbf{2}(t)}\left(\Sigma_{k=0}^{m-1}\mathbf{R}_{\mathbf{k}(t)}\mathbf{\Phi}^{k}\right)}_{\mathbf{G}_{(t)}}\cdot\left(\underbrace{\mathbf{\Phi}^{m}}_{\mathbf{H}_{(t)}}\right)^{i}\right]\mathbf{f}_t = \\
&= -\mathbf{L}_{(t)}f_t
\end{aligned} \tag{30}$$

Following Klein (2000), we calculate the elements of $\mathbf{L}_{(t)}$ using the vectorization operator:[13]

$$vec\left(\mathbf{L}_{(t)}\right) = \left[\mathbf{I} - \mathbf{H}^{T}_{(t)} \otimes \mathbf{F}_{(t)}\right]^{-1} vec\left(\mathbf{G}_{(t)}\right) \tag{31}$$

The existence of the infinite sum stems from (i) fulfilled assumptions of the Blanchard-Kahn theorem (exactly all unstable generalized eigenvalues of **A** and **B** concentrated in the partition (2,2) of matrices **S** and **T**) as well as (ii) stability of the process (29) (eigenvalues of $\mathbf{\Phi}$ lower than 1 in absolute terms). Substitute (30) into (23) after premultiplication by $\mathbf{Z}_{(t)}$ and conformable partitioning:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{\mathbf{11}(t)} & \mathbf{Z}_{\mathbf{12}(t)} \\ \mathbf{Z}_{\mathbf{21}(t)} & \mathbf{Z}_{\mathbf{22}(t)} \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{x}}_{\mathbf{1},t} \\ -\mathbf{L}_{(t)}\mathbf{f}_t \end{bmatrix} \tag{32}$$

[12] Note that a white noise disturbance immediately simplifies (28) to a linear dependence of $\tilde{\mathbf{x}}_{\mathbf{2},t}$ on current $\mathbf{f}_t$.

[13] Premultiply $\mathbf{L}_{(t)}$ by $\mathbf{F}_{(t)}$ and postmultiply by $\mathbf{H}_{(t)}$ so that $\mathbf{F}_{(t)}\mathbf{L}_{(t)}\mathbf{H}_{(t)} = \sum_{i=0}^{+\infty}\mathbf{F}^{i+1}_{(t)}\mathbf{G}_{(t)}\mathbf{H}^{i+1}_{(t)} = \sum_{i=1}^{+\infty}\mathbf{F}^{i}_{(t)}\mathbf{G}_{(t)}\mathbf{H}^{i}_{(t)}$. Note that the only difference between this sum and $\mathbf{L}_{(t)}$ is the first component $\mathbf{G}_{(t)}$, so $\mathbf{L}_{(t)} - \mathbf{F}_{(t)}\mathbf{L}_{(t)}\mathbf{H}_{(t)} = \mathbf{G}_{(t)}$. Vectorize both sides and use the matrix identity $vec(\mathbf{ABC}) = \left(\mathbf{C}^T \otimes \mathbf{A}\right) vec(\mathbf{B})$ to get $vec\left(\mathbf{L}_{(t)}\right) - \mathbf{H}^{T}_{(t)} \otimes \mathbf{F}_{(t)} \cdot vec\left(\mathbf{L}_{(t)}\right) = vec\left(\mathbf{G}_{(t)}\right)$. This can be premultiplied by $\left[\mathbf{I} - \mathbf{H}^{T}_{(t)} \otimes \mathbf{F}_{(t)}\right]^{-1}$, unless the matrix is singular.

After solving out $\tilde{\mathbf{x}}_{\mathbf{1},t}$ from (32), we obtain a linear relationship linking $\mathbf{x}_{\mathbf{1},t}$, $\mathbf{x}_{\mathbf{2},t}$ and $\mathbf{f}_t$:

$$\mathbf{x}_{\mathbf{2},t} = \mathbf{Z}_{\mathbf{21}(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)}\mathbf{x}_{\mathbf{1},t} + \left(\mathbf{Z}_{\mathbf{21}(\mathbf{t})}\mathbf{Z}^{-1}_{\mathbf{11}(t)}\mathbf{Z}_{\mathbf{12}(t)} - \mathbf{Z}_{\mathbf{22}(t)}\right)\mathbf{L}_{(t)}\mathbf{f}_t \tag{33}$$

We exploit the predeterminacy of $\mathbf{x}_{\mathbf{1},t}$ to get:

$$\begin{aligned} E_t\left(\mathbf{x}_{t+1}\right) &= E_t\begin{pmatrix}\mathbf{x}_{\mathbf{1},t+1}\\ \mathbf{x}_{\mathbf{2},t+1}\end{pmatrix} = \\ &= \begin{bmatrix}\mathbf{x}_{\mathbf{1},t+1}\\ \mathbf{Z}_{\mathbf{21}(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)}\mathbf{x}_{\mathbf{1},t+1} + \left(\mathbf{Z}_{\mathbf{21}(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)}\mathbf{Z}_{\mathbf{12}(t+1)} - \mathbf{Z}_{\mathbf{22}(t+1)}\right)\mathbf{L}_{(t+1)}\underbrace{E_t\mathbf{f}_{t+1}}_{\mathbf{\Phi} f_t}\end{bmatrix} = \\ &= \begin{bmatrix}\mathbf{I}\\ \mathbf{Z}_{\mathbf{21}(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)}\end{bmatrix}\mathbf{x}_{\mathbf{1},t+1} + \begin{bmatrix}0\\ \left(\mathbf{Z}_{\mathbf{21}(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)}\mathbf{Z}_{\mathbf{12}(t+1)} - \mathbf{Z}_{\mathbf{22}(t+1)}\right)\mathbf{L}_{(t+1)}\mathbf{\Phi}\end{bmatrix}\mathbf{f}_t \end{aligned} \tag{34}$$

Using (33), we can also replace $\mathbf{x}_{\mathbf{2},t}$ in $\mathbf{x}_t$:

$$\begin{aligned} \mathbf{x}_t &= \begin{bmatrix}\mathbf{x}_{\mathbf{1},t}\\ \mathbf{x}_{\mathbf{2},t}\end{bmatrix} = \begin{bmatrix}\mathbf{x}_{\mathbf{1},t}\\ \mathbf{Z}_{\mathbf{21}(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)}\mathbf{x}_{\mathbf{1},t} + \left(\mathbf{Z}_{\mathbf{21}(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)}\mathbf{Z}_{\mathbf{12}(t)} - \mathbf{Z}_{\mathbf{22}(t)}\right)\mathbf{L}_{(t)}\mathbf{f}_t\end{bmatrix} = \\ &= \begin{bmatrix}\mathbf{I}\\ \mathbf{Z}_{\mathbf{21}(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)}\end{bmatrix}\mathbf{x}_{\mathbf{1},t} + \begin{bmatrix}0\\ \left(\mathbf{Z}_{\mathbf{21}(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)}\mathbf{Z}_{\mathbf{12}(t)} - \mathbf{Z}_{\mathbf{22}(t)}\right)\mathbf{L}_{(t)}\end{bmatrix}\mathbf{f}_t \end{aligned} \tag{35}$$

In the example considered here, the vector of predetermined variables $\mathbf{x}_{\mathbf{1},t}$ contains lags of all elements in $\mathbf{x}_{\mathbf{2},t}$. Accordingly, some rows in $\mathbf{A}_{(t)}$, $\mathbf{B}_{(t)}$ and $\mathbf{C}_{(t)}$ were trivial identities defining the equivalence between some elements of $\mathbf{x}_{\mathbf{1},t+1}$ and $\mathbf{x}_{\mathbf{2},t}$. With the relation between $\mathbf{x}_{\mathbf{1},t}$ and $\mathbf{x}_{\mathbf{2},t}$ in hand, we can drop these rows and denote the remaining matrices as $\overline{\mathbf{A}}_{(t)}$, $\overline{\mathbf{B}}_{(t)}$ and $\overline{\mathbf{C}}_{(t)}$. Rewrite (15) without these rows, using (34) and (35):

$$\overline{\mathbf{A}}_{(t)} \begin{bmatrix} \mathbf{I} \\ \mathbf{Z_{21}}_{(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)} \end{bmatrix} \mathbf{x}_{\mathbf{1},t+1} + \overline{\mathbf{A}}_{(t)} \begin{bmatrix} 0 \\ \left(\mathbf{Z_{21}}_{(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)}\mathbf{Z_{12}}_{(t+1)} - \mathbf{Z_{22}}_{(t+1)}\right)\mathbf{L}_{(t+1)}\mathbf{\Phi} \end{bmatrix} \mathbf{f}_t =$$
$$= \overline{\mathbf{B}}_{(t)} \begin{bmatrix} \mathbf{I} \\ \mathbf{Z_{21}}_{(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)} \end{bmatrix} \mathbf{x}_{\mathbf{1},t} + \overline{\mathbf{B}}_{(t)} \begin{bmatrix} 0 \\ \left(\mathbf{Z_{21(t)}}\mathbf{Z^{-1}_{11(t)}}\mathbf{Z_{12(t)}} - \mathbf{Z_{22(t)}}\right)\mathbf{L_{(t)}} \end{bmatrix} \mathbf{f}_t + \overline{\mathbf{C}}\mathbf{f}_t \tag{36}$$

The solution of (36) with respect to $\mathbf{x}_{1,t+1}$ is the searched law of motion of the form (14):

$$\mathbf{x}_{\mathbf{1},t+1} = \left(\overline{\mathbf{A}}_{(t)} \begin{bmatrix} \mathbf{I} \\ \mathbf{Z_{21}}_{(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)} \end{bmatrix}\right)^{-1} \overline{\mathbf{B}}_{(t)} \begin{bmatrix} \mathbf{I} \\ \mathbf{Z_{21}}_{(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)} \end{bmatrix} \mathbf{x}_{\mathbf{1},t} +$$
$$+ \left(\overline{\mathbf{A}}_{(t)} \begin{bmatrix} \mathbf{I} \\ \mathbf{Z_{21}}_{(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)} \end{bmatrix}\right)^{-1} \cdot$$
$$\cdot \left(\overline{\mathbf{C}} + \overline{\mathbf{B}}_{(t)} \begin{bmatrix} 0 \\ \left(\mathbf{Z_{21}}_{(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)}\mathbf{Z_{12}}_{(t)} - \mathbf{Z_{22}}_{(t)}\right)\mathbf{L}_{(t)} \end{bmatrix} - \overline{\mathbf{A}}_{(t)} \begin{bmatrix} 0 \\ \left(\mathbf{Z_{21}}_{(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)}\mathbf{Z_{12}}_{(t+1)} - \mathbf{Z_{22}}_{(t+1)}\right)\mathbf{L}_{(t+1)}\mathbf{\Phi} \end{bmatrix}\right) \mathbf{f}_t \tag{37}$$

3.3 Model with heterogenous information sets

We use the method of undetermined coefficients proposed by Christiano (2002) to solve the model with heterogenous information sets of economic agents used for forming their expectations in individual countries of the monetary union.[14] This method ascribes an individual information set to every equation in the system and expectational terms in any equation are conditional upon the content of its own information set. Every information set contains all the past values of the random disturbances and part of its contemporaneous values (in an extreme case: all or none of them).

Christiano (2002) solves a linear dynamic rational expectations model of the form:

$$\varepsilon_t \left(\sum_{i=0}^{r} \boldsymbol{\alpha}_\mathbf{i} \mathbf{z}_{t+r-1-i} + \sum_{i=0}^{r-1} \boldsymbol{\beta}_\mathbf{i} \mathbf{s}_{t+r-1-i} \right) = 0 \tag{38}$$

with $\mathbf{z}_t = \begin{bmatrix} \mathbf{z}_{\mathbf{1},t} & \mathbf{z}_{\mathbf{2},t} \end{bmatrix}^T$, $\mathbf{z}_{\mathbf{1},t}$ - n_1-dimensional vector of endogenous variables non-predetermined at t, $\mathbf{z}_{\mathbf{2},t}$ contains q lags of $\mathbf{z}_{\mathbf{1},t}$ necessary to determine $\mathbf{z}_{\mathbf{1},t+1}$ at $t+1$. In our model, the lag length

[14] A detailed description of the method in full generality can be found in Christiano (2002). For the sake of presentational simplicity, this section consumes all the possible simplifications in the case that we are investigating.

does not exceed 1, which means $q = 0$ and $\mathbf{z}_t = \mathbf{z}_{1,t} = \begin{bmatrix} 1 & \mathbf{P}_t & i_t & \mathbf{y}_t & \boldsymbol{\pi}_t \end{bmatrix}^T$ is a vector of length $\tilde{n} \equiv 2 + 3n$ (n – number of countries).

However, equation (38) is not equivalent to (13) due to a conceptual difference in expectation operators. When information sets for individual equations are heterogenous, $\varepsilon_t(.)$ denotes rational expectations based on an equation-specific (i.e. country-specific) information set:

$$\epsilon_t(\mathbf{x_{t+1}}) = \begin{bmatrix} E_t(\mathbf{x_{1}}_{,t+1}|\Omega_{1,t}) \\ E_t(\mathbf{x_{2}}_{,t+1}|\Omega_{2,t}) \\ \vdots \\ E_t(\mathbf{x_{n}}_{,t+1}|\Omega_{n,t}) \end{bmatrix} \tag{39}$$

At t, only the union's central bank is familiar with the entire vector of current country-specific demand and supply disturbances, $\mathbf{f}_t = \begin{bmatrix} \varepsilon_{y,1,t} & \ldots & \varepsilon_{y,n,t} & \varepsilon_{\pi,1,t} & \ldots & \varepsilon_{\pi,n,t} \end{bmatrix}^T$. The only contemporaneous values of shocks that economic agents in country j take into account are ones concerning their own country, i.e. $\varepsilon_{y,j,t}$ and $\varepsilon_{\pi,j,t}$. Shocks to the other economies enter the information set of country j agents with a one period lag.

The restrictions excluding some elements of $\mathbf{f}_t$ from some equations' information sets are summarized in the matrix $\boldsymbol{\tau}$ sized $2n \times \tilde{n}$. Its columns correspond with equations in the system, rows – with elements of $\mathbf{f}_t$; $\tau_{[i,j]} = 1$ when i-th innovation is included in the information set of equation j and $\tau_{[i,j]} = 0$ otherwise. In our setup, $\boldsymbol{\tau} = \begin{bmatrix} \mathbf{0}_{n\times 2n+1} & \mathbf{I}_n & \mathbf{I}_n & \mathbf{1}_{n\times 1} \\ \mathbf{0}_{n\times 2n+1} & \mathbf{I}_n & \mathbf{I}_n & \mathbf{1}_{n\times 1} \end{bmatrix}$.

In the framework of Christiano, we need to expand the random vector by its first lag: $\mathbf{s}_t = \begin{bmatrix} \mathbf{f}_t \\ \mathbf{f}_{t-1} \end{bmatrix}$ exactly due to the exclusion restrictions in τ. Like in Subsection 3.2, we assume a VAR respresentation (29) for $\mathbf{f}_t$. This method, however, additionally requires the knowledge of the variance-covariance matrix of $\boldsymbol{\varepsilon}$: $\mathbf{V_e} = E\left(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T\right) = \begin{bmatrix} \boldsymbol{\Sigma}_\mathbf{d} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_\mathbf{s} \end{bmatrix}$ with independence of demand and supply disturbances assumed.[15] This implies the following VAR representation for $\mathbf{s}_t$:

$$\begin{bmatrix} \mathbf{f}_t \\ \mathbf{f}_{t-1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{\Phi} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}}_{\mathbf{P}} \begin{bmatrix} \mathbf{f}_{t-1} \\ \mathbf{f}_{t-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ \mathbf{0} \end{bmatrix} \tag{40}$$

Knowing the matrices $\boldsymbol{\alpha_0}, \boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}, \boldsymbol{\beta_0}, \boldsymbol{\beta_1}$ and $\boldsymbol{\tau}$, $\mathbf{V_e}$, $\mathbf{P}$ (see Appendix 2), we can apply the method of Christiano (2002) to compute the matrices $\mathbf{M}, \mathbf{N}$ such that the solution to (38) is of the form

$$\mathbf{z}_t = \mathbf{M}\mathbf{z}_{t-1} + \mathbf{N}\mathbf{s}_t \tag{41}$$

Under complete information sets, the following framework is fully equivalent to the standard methods (Blanchard-Kahn, Klein, Sims or Uhlig). Moreover, $\mathbf{M}$ is always the same as derived via standard methods, i.e. independent on (in)completeness of information sets. When at least one of the information sets is incomplete, the key step in pinning down $\mathbf{N}$ is an orthogonal projection from the space of random disturbances included in an equation's information set (where $\mathbf{f}_{t-1}$ and part of $\mathbf{f}_t$ jointly belong) to the space of all contemporanous and lagged random disturbances (where $\mathbf{s}_t$ belongs). To obtain $\mathbf{M}$, merge $\mathbf{z_t}$ and $\mathbf{z}_{t-1}$ and write the system (38) skipping $\mathbf{s}_t$:

$$\underbrace{\begin{bmatrix} \boldsymbol{\alpha_0} & \mathbf{0}_{\tilde{n}\times\tilde{n}} \\ \mathbf{0}_{\tilde{n}\times\tilde{n}} & \mathbf{I_{\tilde{n}}} \end{bmatrix}}_{\mathbf{a}} \begin{bmatrix} \mathbf{z}_{t+1} \\ \mathbf{z}_t \end{bmatrix} + \underbrace{\begin{bmatrix} \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} \\ -\mathbf{I_{\tilde{n}}} & \mathbf{0}_{\tilde{n}\times\tilde{n}} \end{bmatrix}}_{\mathbf{b}} \begin{bmatrix} \mathbf{z}_t \\ \mathbf{z}_{t-1} \end{bmatrix} = \mathbf{0} \tag{42}$$

Factorize $\mathbf{a}$ and $\mathbf{b}$ using the generalized Schur decomposition (as in (16)). Arrange matrices $\mathbf{Q}, \mathbf{Z}, \mathbf{S}, \mathbf{T}$ so that the zeros on the main diagonal of $\mathbf{S}$ are located in its lower-right portion. This separates upper-left portions of $\mathbf{S}$ and $\mathbf{T}$ denoted $\mathbf{S_{11}}$ and $\mathbf{T_{11}}$ respectively. Partition conformably $\mathbf{Z}^H = \begin{bmatrix} \mathbf{Z_1}^H \\ \mathbf{Z_2}^H \end{bmatrix}$. Then, use the eigenvalue-eigenvector decomposition:

$$-\mathbf{S_{11}^{-1}}\mathbf{T_{11}} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1} \tag{43}$$

Let $\tilde{\mathbf{P}}$ denote the rows of $\mathbf{P}^{-1}$ corresponding to the eigenvalues exceeding 1 in absolute terms. Let

[15] Christiano (2002) emphasizes that this input is only required in the presence of at least one non-empty and incomplete information set.

$\mathbf{D} = \begin{bmatrix} \tilde{\mathbf{P}}\mathbf{Z_1^H} \\ \mathbf{Z_2^H} \end{bmatrix} = \begin{bmatrix} \underbrace{\mathbf{D_1}}_{\tilde{n}} & \underbrace{\mathbf{D_2}}_{\tilde{n}} \end{bmatrix}$. Matrix $\mathbf{M}$ is finally obtained as lower $\tilde{n}$ rows of the matrix $-\mathbf{D_1^{-1}D_2}$.

To find $\mathbf{N}$, define for every equation i (i.e. for every column $\boldsymbol{\tau}_{[:,i]}$ in $\boldsymbol{\tau}$) matrix $\mathbf{R_i}$ as unity matrix in which the rows corresponding to zeros in the vector $\boldsymbol{\tau}_{[:,i]}$ were dropped. The orthogonal projection mentioned before implies matrices $\mathbf{C} = \sum_{i=0}^{\infty} \mathbf{\Phi}^i \mathbf{V_e} \left(\mathbf{\Phi}^T\right)^i$, $\boldsymbol{\varphi}_i = \begin{bmatrix} \mathbf{R_i C R_i^T} & \mathbf{R_i \Phi C} \\ \mathbf{C}^T \mathbf{\Phi}^T \mathbf{R}_i^T & \mathbf{C} \end{bmatrix}$, $\boldsymbol{\phi}_i = \begin{bmatrix} \mathbf{C R_i^T} & \mathbf{\Phi C} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{a_i} & \mathbf{a}_{\mathbf{i}\boldsymbol{\theta}} \end{bmatrix} = \boldsymbol{\phi}_\mathbf{i} \boldsymbol{\varphi}_\mathbf{i}^{-1}$, whereby the number of columns in $\mathbf{a_i}$ equals the number of nonzero elements in $\boldsymbol{\tau}_{[:,i]}$. Let $\mathbf{R} = \left[\begin{array}{cc|cc|c|cc} (\mathbf{a_1 R_1})^T & \mathbf{0} & \mathbf{0} & & \dots & \mathbf{0} & \\ \mathbf{a}_{\mathbf{1}\boldsymbol{\theta}}^T & \mathbf{I} & & & & & \\ \hline \mathbf{0} & & (\mathbf{a_2 R_2})^T & \mathbf{0} & \dots & \mathbf{0} & \\ & & \mathbf{a}_{\mathbf{2}\boldsymbol{\theta}}^T & \mathbf{I} & & & \\ \hline \vdots & & \vdots & & \ddots & \vdots & \\ \hline \mathbf{0} & & \mathbf{0} & & \dots & (\mathbf{a_{\tilde{n}} R_{\tilde{n}}})^T & \mathbf{0} \\ & & & & & \mathbf{a}_{\tilde{\mathbf{n}}\boldsymbol{\theta}}^T & \mathbf{I} \end{array}\right]$, and $\tilde{\mathbf{R}}$ be defined as $\mathbf{R}$ with dropped zero rows resulting from zero elements in τ. Let $\tilde{d} = \tilde{R} vec\left[P^T \beta_0^T + \beta_1^T\right]$, and let $\tilde{q}$ be defined as matrix $\tilde{\mathbf{R}}\left(\boldsymbol{\alpha_0} \otimes \mathbf{P}^T + (\boldsymbol{\alpha_0}\mathbf{A} + \boldsymbol{\alpha_1}) \otimes \mathbf{I}\right)$ in which the columns were dropped whose numbers corresponded to the rows in $\tilde{R}$ that we had dropped before. The elements of $\mathbf{N}$ are defined by the relationship $vec\left(\mathbf{N}^T\right) = -\tilde{\mathbf{q}}^{-1}\tilde{\mathbf{d}}$, whereby the left-hand side vector – before de-vectorization into $\mathbf{N}^T$ – needs to be widened and filled with zeros at the indices of dropped rows in $\tilde{\mathbf{R}}$ (dropped columns in $\tilde{\mathbf{q}}$).

4 Simulation results

The model described in Section 2 and solved with the methods from Section 3 has beed used to simulate the path of the output gap and inflation rate with different assumptions regarding:

1. the home bias of the ECB Governing Council members (α);

2. the information set underlying the formation of expectations of future output and inflation in the countries of a monetary union.

Table 1 contains a set of model parameters used in the simulations. For simplicity, we assume homogenous parameters of the IS and Phillips curves across countries. Parameter values are median values of statistically significant estimates among 12 euro area countries over the time span 1999-2008 (from Torój, 2009). The parameters, following the dominant empirical approach in the New Keynesian literature, were estimated via generalized method of moments (see Galí and Gertler, 1999; Galí et al., 2001; Goodhart and Hofmann, 2005) with standard instrument sets for both curves. The parameters for the Taylor rule and AR processes of the random disturbances are parametrized as in the literature overview by Lindé (2005).

Table 1: Parameters of the simulated model

ω_f	0.55	β_r	0.09	γ_π	1.5	ρ_π	0.1
ω_b	0.45	β_c	0.04	γ_y	0.5	ρ_y	0.5
β_f	0.5	β_s	0.09	ρ	0.5		
β_b	0.5	γ	0.05				
Source: Torój (2009)				Source: Lindé (2005)			

Every pair of variances compared below results from a path of variables generated with the same path of 10000 random shocks. Demand and supply disturbances were assumed to be independent. The variances of individual shocks were calibrated in such a way that the second moments of the baseline paths match those observed in the data on inflation and output.

The results generally confirm those obtained from a purely backward-looking model by Kosior et al. (2008), at least on the qualitative level. The rotation in the ECB Governing Council, coupled with some home bias in interest rate decisions among its members, can boost the variance of inflation and output gap. Table 2 presents the results of simulation when a monetary union consists of 4 equally sized countries and 2 country representatives participate in every vote. The cycle of rotation lasts 8 quarters, and there is a switch every 2 quarters. The standard deviations of output gap and inflation rise as α increases. For $\alpha = 0.5$, the standard deviation of the output gap is 0.42% higher than for $\alpha = 0$ (i.e. in the model with constant parameters). The standard deviation of inflation rises analogously by 1.64%.

Table 2: S.D. of output gap and inflation (expressed as a share of S.D. under baseline scenario)

	y	π
α	**w=0,25**	**w=0,25**
0	1.0000	1.0000
0.1	1.0003	1.0011
0.2	1.0008	1.0033
0.3	1.0017	1.0066
0.4	1.0028	1.0110
0.5	1.0042	1.0164

When the country sizes differ, so do the results for big, mid-size and small economies. Table 3 presents the results of simulations generated with a 4-country model of monetary union with relative country sizes of 0.4, 0.3, 0.2 and 0.1. Once again, 2 country representatives vote at a time, the rotation cycle is 8 quarters long and the right of vote is granted to the governors 7, 5, 3 or 1 time a cycle, in line with their country size.

It is only in the largest economy that the output gap volatility slightly declines as α rises. The country of size 0.3 enjoys a slightly positive α for the same reason. However, starting from approximately $\alpha = 0.3$, it suffers from a rise in output volatility as the impact of „imported" instability from two small economies and the relative loss of the central bank's focus in favour of the greatest country begin to dominate. In the case of the smallest economies, this effect is visible for any positive home bias because their relative weight in the union-wide Taylor rule declines with growing α (it is more efficient to make big neighbours „pro-european" than to remain small and home-biased) and because they import each other's volatility simultaneaously. With $\alpha = 0.5$, the standard deviation of the output gap is ca. $0.76 - 0.77\%$ higher and inflation – ca. 2% higher than in the baseline scenario.

Table 3: S.D. of output gap and inflation (expressed as ratio to S.D. under baseline scenario); different country sizes

	y				π			
	w=				w=			
α	**0.4**	**0.3**	**0.2**	**0.1**	**0.4**	**0.3**	**0.2**	**0.1**
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	0.9985	0.9999	1.0012	1.0012	0.9961	1.0001	1.0031	1.0028
0.2	0.9971	0.9999	1.0026	1.0026	0.9927	1.0008	1.0066	1.0062
0.3	0.9958	1.0001	1.0041	1.0041	0.9901	1.0020	1.0107	1.0100
0.4	0.9947	1.0004	1.0058	1.0058	0.9881	1.0038	1.0152	1.0145
0.5	0.9937	1.0008	1.0076	1.0077	0.9867	1.0062	1.0201	1.0195

The impact of excluding the contemporaneous values of foreign demand and supply shocks from the information set of domestic agents is vague, even on the qualitative level. Table 4 is composed of standard deviations of output gaps when the information sets are incomplete, expressed as shares of standard deviations in the baseline scenario with complete information. Depending on the correlation of shocks between countries, serial correlation of country-specific shocks and – possibly – the country size, the incompleteness of the information set raises or reduces the volatility of output.

When the serial correlation of demand shocks is low, a home-biased information set reduces the variance of the output gap. Foreign demand shocks with low persistence have only limited impact on the domestic economy and start to influence domestic expectations once they have partially been absorbed. Note that the serial correlation of demand shocks at $0-0.2$ remains far lower than the empirical evidence would suggest (see Lindé, 2005).

Incomplete information sets also reduce the output volatility when the synchronicity of shocks between countries is high. In such a monetary union, the country-specific information set is sufficient to approximate some of the foreign noise and support the expectations as an auxiliary adjustment channel. When agents believe that shocks are correlated across countries, they do not fear that an asymmetric shock would induce inadequacy of the common interest rate and domestic macroeconomic aggregates.

On the other hand, under a high persistence and low symmetry of demand shocks, heterogenous and incomplete information sets yield higher variance of the output. Only with a lag does highly useful

Table 4: Heterogenous information sets – S.D. of output gap (expressed as ratio to S.D. under homogenous information sets)

cross-country correlation of demand shocks	serial correlation of demand shocks	country size				
		0.4	**0.3**	**0.2**	**0.07**	**0.03**
0	0	0.78	0.78	0.77	0.76	0.76
	0.2	0.91	0.91	0.91	0.90	0.90
	0.4	**1.13**	**1.13**	**1.13**	**1.14**	**1.14**
	0.6	**1.50**	**1.52**	**1.54**	**1.58**	**1.58**
	0.8	**2.22**	**2.30**	**2.44**	**2.52**	**2.57**
0.2	0	0.72	0.72	0.71	0.71	0.71
	0.2	0.85	0.85	0.84	0.84	0.84
	0.4	**1.05**	**1.05**	**1.06**	**1.05**	**1.05**
	0.6	**1.39**	**1.42**	**1.44**	**1.45**	**1.45**
	0.8	**2.02**	**2.11**	**2.20**	**2.30**	**2.32**
0.4	0	0.51	0.50	0.49	0.49	0.49
	0.2	0.60	0.59	0.59	0.58	0.57
	0.4	0.75	0.74	0.74	0.72	0.71
	0.6	0.99	0.99	**1.01**	**1.01**	0.99
	0.8	**1.48**	**1.55**	**1.55**	**1.59**	**1.64**
0.6	0	0.23	0.22	0.21	0.20	0.19
	0.2	0.28	0.26	0.26	0.24	0.23
	0.4	0.34	0.33	0.31	0.31	0.29
	0.6	0.47	0.45	0.43	0.42	0.40
	0.8	0.70	0.71	0.69	0.69	0.68
0.8	0	0.10	0.09	0.07	0.06	0.06
	0.2	0.11	0.10	0.08	0.07	0.06
	0.4	0.13	0.12	0.10	0.08	0.08
	0.6	0.16	0.14	0.13	0.10	0.10
	0.8	0.19	0.18	0.16	0.14	0.13

information arrive in agent's expectations. This hampers the „expectations" channel of adjustment and the stabilization of output around the potential level is more protracted, which generates a higher volatility. Note that such a stochastic environment is a contradiction of what the optimum currency area theory views as perfect (synchronized business cycles and at most temporary shocks).

Finally, note the effect in the row of Table 4 corresponding to empirically plausible values of demand shocks' serial correlation equal 0.6 and cross-country correlation equal 0.4. It suggests that in the mid-size economies, a limited information set might generate higher output volatility, whereas in big and small economies – lower volatility. Although this result seems to be quantitatively limited, the very fact that this model was capable to reproduce it might be seen as a weak confirmation of some tentative evidence reported in earlier literature. Namely, big economies benefit mainly from the stabilizing effects of common monetary policy and small ones – from the competitiveness channel. At the same time, expectation as a supportive channel of stabilization after asymmetric shocks could be particularly useful in mid-size member countries of a monetary union. In such countries, economic agents must therefore carefully watch the external environment. This weak implication of the model certainly needs further research, as it might be of particular importance for countries such as Poland.

5 Conclusions

This paper generalizes the analytical methods of solving linear rational expectations models to the case of time-varying, nonstochastic parameters. The assumption of a finite cycle in which the parameter values recur is thereby exploited. The solution is exemplified with the case of autoregressive random disturbances. We also apply the method of Christiano (2002) to introduce heterogeneity in individual countries' information sets.

The simulations based on the former solution method confirm the previous findings from the literature: the rotation in the ECB Governing Council, as implemented by the Treaty of Nice, coupled with home bias in interest rate decisions taken by the members of the Council, increases the volatility of output and inflation in most of the small and mid-size economies. However, the rise in standard deviation – with the parametrisation considered here – is very limited and amounts to a maximum of 2% in small economies.

Forming expectations on the country level with a partial, home-biased information set may in turn lead to a rise or a decline in the output volatility at home, depending on (i) the serial correlation of demand disturbances, (ii) their correlation across countries and possibly (iii) the country size. When the properties of shocks do not conform to the optimum currency area theory, i.e. they are hardly synchronized and highly persistent, the output volatility can rise as a result of introducing incomplete information sets. There is also some limited evidence that this effect could particularly occur in mid-size economies, where expectations stabilize the output to a higher extent than common monetary policy and competitiveness channel, unlike in big or small economies respectively.

A number of questions arise for further research. First of all, heterogeneity of expectations clearly interacts with other aspects of euro area heterogeneity, such as market rigidities or inflation persistence. The inclusion of structural stress into the analysis of ECB rotation framework would be possible when a profounded research on long-run inflation and natural interest rate differentials within the euro area were carried out. Also, it would be interesting to use the solution by Christiano (2002) to derive an empirical test for the size of agent's information set in the economy of a real euro area member country.

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Appendix 1: Construction of matrices A, B and C in Subsection 3.2

$$\mathbf{A}=\begin{bmatrix} 1 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & 0 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{I}_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{I}_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{I}_n & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ 0 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & 1 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \boldsymbol{c_y} & -\boldsymbol{\beta_c} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & -\boldsymbol{\beta}_{\mathbf{r}} & \boldsymbol{\beta_f} & \boldsymbol{\beta_r} \\ \boldsymbol{c_\pi} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \boldsymbol{\omega_f} \end{bmatrix},$$

$$\mathbf{B}_{(\mathbf{t})}{}^{\mathrm{T}}=\begin{bmatrix} 1 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & (1-\rho)\left(r^*+\pi^*-\gamma_\pi\pi^*\right) & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{I}_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & -\boldsymbol{\beta_b} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & -\boldsymbol{\omega_b} \\ 0 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \rho & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times n} & 0.25\cdot\mathbf{I}_n & \mathbf{I}_n & \mathbf{0}_{n\times n} & (1-\rho)\,\gamma_y\left((1-\alpha)\,\mathbf{w}^T+\alpha\boldsymbol{a}_t^T\right) & \boldsymbol{\beta}_{\mathbf{s}} & -\boldsymbol{\gamma} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{I}_n & (1-\rho)\,\gamma_\pi\left((1-\alpha)\,\mathbf{w}^T+\alpha\boldsymbol{a}_t^T\right) & \mathbf{0}_{n\times n} & \mathbf{I}_n \end{bmatrix},$$

$$\mathbf{C}=\begin{bmatrix} \mathbf{0}_{3n+1,2n} \\ \boldsymbol{I}_{2n} \end{bmatrix},\ \mathbf{c}_{\boldsymbol{\pi}}=\begin{bmatrix} (1-\omega_{b,1}-\omega_{f,1})\,\pi_1^* \\ (1-\omega_{b,2}-\omega_{f,2})\,\pi_2^* \\ \vdots \\ (1-\omega_{b,27}-\omega_{f,27})\,\pi_n^* \end{bmatrix},\ \mathbf{c}_{\mathbf{y}}=\begin{bmatrix} \beta_{r,1}r_1^* \\ \beta_{r,2}r_2^* \\ \vdots \\ \beta_{r,27}r_n^* \end{bmatrix},$$

$$\boldsymbol{\gamma}=diag\begin{bmatrix}\gamma_1 & \gamma_2 & \ldots & \gamma_n\end{bmatrix},\ \boldsymbol{\omega}_{\mathbf{f}}=diag\begin{bmatrix}\omega_{f,1} & \omega_{f,2} & \ldots & \omega_{f,n}\end{bmatrix},$$

$$\boldsymbol{\omega}_{\mathbf{b}}=\begin{bmatrix}\omega_{b,1} & \omega_{b,2} & \ldots & \omega_{b,n}\end{bmatrix},\ \boldsymbol{\beta}_{\mathbf{f}}=diag\begin{bmatrix}\beta_{f,1} & \beta_{f,2} & \ldots & \beta_{f,n}\end{bmatrix},$$

$$\boldsymbol{\beta}_{\mathbf{b}}=diag\begin{bmatrix}\beta_{b,1} & \beta_{b,2} & \ldots & \beta_{b,n}\end{bmatrix},\ \boldsymbol{\beta}_{\mathbf{r}}=diag\begin{bmatrix}\beta_{r,1} & \beta_{r,2} & \ldots & \beta_{r,n}\end{bmatrix},$$

$$\boldsymbol{\beta}_{\mathbf{s}}=\begin{bmatrix} 1 & -\beta_{s,1}\frac{w_2}{1-w_1} & \ldots & -\beta_{s,1}\frac{w_n}{1-w_1} \\ -\beta_{s,2}\frac{w_1}{1-w_2} & 1 & \ldots & -\beta_{s,2}\frac{w_n}{1-w_2} \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_{s,n}\frac{w_1}{1-w_n} & -\beta_{s,n}\frac{w_2}{1-w_n} & \ldots & 1 \end{bmatrix},\ \boldsymbol{\beta}_{\mathbf{c}}=\begin{bmatrix} \beta_{c,1} & -\beta_{c,1}\frac{w_2}{1-w_1} & \ldots & -\beta_{c,1}\frac{w_n}{1-w_1} \\ -\beta_{c,2}\frac{w_1}{1-w_2} & \beta_{c,2} & \ldots & -\beta_{c,2}\frac{w_n}{1-w_2} \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_{c,n}\frac{w_1}{1-w_n} & -\beta_{c,n}\frac{w_2}{1-w_n} & \ldots & \beta_{c,n} \end{bmatrix}.$$

Appendix 2: Construction of matrices used in Subsection 3.3

$$r = 2,\ \boldsymbol{\alpha_0} = \begin{bmatrix} \mathbf{0}_{n+2\times n+2} & \mathbf{0}_{n+2\times n} & \mathbf{0}_{n+2\times n} \\ \mathbf{0}_{n\times n+2} & \boldsymbol{\beta}_{\mathbf{f}} & \boldsymbol{\beta}_{\mathbf{r}} \\ \mathbf{0}_{n\times n+2} & \mathbf{0}_{n\times n} & \boldsymbol{\omega}_{\mathbf{f}} \end{bmatrix},$$

$$\boldsymbol{\alpha_1} = \begin{bmatrix} 1 & \mathbf{0}_{1\times n} & 0 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & -\mathbf{I}_n & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & 0,25\mathbf{I}_n \\ (1-\rho)(r^* + \pi^* - \gamma_\pi \pi^*) & \mathbf{0}_{1\times n} & -1 & (1-\rho)\gamma_y \mathbf{w}^T & (1-\rho)\gamma_\pi \mathbf{w}^T \\ \mathbf{c}_{\mathbf{y}} & -\boldsymbol{\beta}_{\mathbf{c}} & -\boldsymbol{\beta}_{\mathbf{r}} & -\boldsymbol{\beta}_{\mathbf{s}} & \mathbf{0}_{n\times n} \\ \mathbf{c}_{\boldsymbol{\pi}} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \boldsymbol{\gamma} & -\mathbf{I}_n \end{bmatrix},$$

$$\boldsymbol{\alpha_2} = \begin{bmatrix} -1 & \mathbf{0}_{1\times n} & 0 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{I}_n & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ 0 & \mathbf{0}_{1\times n} & \rho & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \boldsymbol{\beta}_{\boldsymbol{b}} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \boldsymbol{\omega}_{\boldsymbol{b}} \end{bmatrix}, \quad \boldsymbol{\beta_0} = \mathbf{0}_{\tilde{n}\times 4n}, \quad \boldsymbol{\beta_1} =$$

$$\begin{bmatrix} \mathbf{0}_{(n+1)\times n} & \mathbf{0}_{(n+1)\times n} & \mathbf{0}_{(n+1)\times 2n} \\ \mathbf{I}_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 2n} \\ \mathbf{0}_{n\times n} & \mathbf{I}_n & \mathbf{0}_{n\times 2n} \\ \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times 2n} \end{bmatrix}.$$